

# ON THE UNIPOTENT CHARACTERS OF THE REE GROUPS OF TYPE $G_2$

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**ABSTRACT.** This note is concerned with the unipotent characters of the Ree groups of type  $G_2$ . We determine the roots of unity associated by Lusztig and Digne-Michel to unipotent characters of  ${}^2G_2(3^{2n+1})$  and we prove that the Fourier matrix of  ${}^2G_2(3^{2n+1})$  defined by Geck and Malle satisfies a conjecture of Digne-Michel. Our main tool is the Shintani descent of Ree groups of type  $G_2$ .

## 1. INTRODUCTION

Let  $\mathbf{G}$  be a connected reductive group defined over the finite field  $\mathbb{F}_q$  with  $q$  elements of characteristic  $p > 0$ , and let  $F$  be the corresponding Frobenius map. Let  $\mathbf{T}$  be a maximal rational torus of  $\mathbf{G}$ , contained in a rational Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$ . We denote by  $W = \mathbf{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$  the Weyl group of  $\mathbf{G}$ . For  $w \in W$ , there is the corresponding Deligne-Lusztig character  $R_w$  of the finite fixed-point subgroup  $\mathbf{G}^F$ . We refer to [4, §7.7] for a precise construction. Then we define the set of unipotent characters of  $\mathbf{G}^F$  by

$$\mathcal{U}(\mathbf{G}^F) = \{\chi \in \text{Irr}(\mathbf{G}^F) \mid \langle \chi, R_w \rangle_{\mathbf{G}^F} \neq 0 \text{ for some } w \in W\}.$$

Lusztig [9] and Digne-Michel [5] associated to every  $\chi \in \mathcal{U}(\mathbf{G}^F)$  a root of unity  $\omega_\chi \in \overline{\mathbb{Q}_\ell}$  (where  $\overline{\mathbb{Q}_\ell}$  is the  $\ell$ -adic field for  $\ell \neq p$ ) as follows. We denote by  $\delta$  the order of the automorphism of  $W$  induced by  $F$ . For  $w \in W$ , we denote by  $n_w \in \mathbf{N}_{\mathbf{G}}(\mathbf{T})$  an element such that  $n_w \mathbf{T} = w$ . We set  $X_w = \{x\mathbf{B} \mid x^{-1}F(x) \in \mathbf{B}n_w\mathbf{B}\}$  the corresponding Deligne-Lusztig variety. For every integer  $j$  we denote by  $H_c^j(X_w, \overline{\mathbb{Q}_\ell})$  the  $j$ -th  $\ell$ -adic cohomology space with compact support over  $\overline{\mathbb{Q}_\ell}$ , associated to  $X_w$ . Therefore, the groups  $\langle F^\delta \rangle$  and  $\mathbf{G}^F$  act on  $X_w$ , which induces linear operations on  $H_c^j(X_w, \overline{\mathbb{Q}_\ell})$ . Hence  $H_c^j(X_w, \overline{\mathbb{Q}_\ell})$  is a  $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -module, and  $F^\delta$  acts on this space as a linear endomorphism. We also fix an eigenvalue  $\lambda$  of  $F^\delta$ , and we denote by  $F_{\lambda,j}$  its generalized eigenspace. Since the actions of  $\langle F^\delta \rangle$  and  $\mathbf{G}^F$  commute, the space  $F_{\lambda,j}$  is a  $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -module. Moreover, the irreducible constituents of  $\mathbf{G}^F$  occurring in this module are unipotent. Conversely, for every unipotent character  $\chi$  of  $\mathbf{G}^F$ , there are  $w \in W$ ,  $\lambda \in \overline{\mathbb{Q}_\ell}^\times$  and  $j \in \mathbb{N}$ , such that  $\chi$  occurs in the character associated to  $F_{\lambda,j}$ . Lusztig [9] and Digne-Michel [5] have shown that  $\lambda = q^{s/2}\omega_\chi$  for some non-negative integer  $s$  and a root of unity  $\omega_\chi \in \overline{\mathbb{Q}_\ell}^\times$  which depends only on  $\chi$ .

The set of roots associated as above to the unipotent characters have been computed by Lusztig in [9] for finite reductive groups if  $F$  is split. Moreover Lusztig computed in [8] the set of roots for the unipotent characters appearing in  $H_c^j(X_{w_{\text{cox}}}, \overline{\mathbb{Q}_\ell})$  where  $w_{\text{cox}}$  denotes the Coxeter element of  $W$ , with no condition on  $F$ . This work is completed for the cases that  $F$  is a non-split Frobenius map by

Geck and Malle in [7]. However, for few cases, the methods of Lusztig and Geck-Malle allow to associate the roots to their unipotent characters only up to complex conjugation. This is for example the case for the Suzuki and the Ree groups. In [1] we remove this indetermination for the unipotent characters of the Suzuki groups.

Moreover, we recall that Lusztig [9] associated to most of sets  $\mathcal{U}(\mathbf{G}^F)$  some non-abelian Fourier matrices, which involve the decomposition in unipotent characters of  $R_w$  for  $w \in W$ .

This note is concerned with the Fourier matrices and the roots of unity, associated as above to the unipotent characters of the Ree group  ${}^2G_2(q)$  for  $q = 3^{2n+1}$ . For these groups, the method in [9] does not allow to define Fourier matrices. However using the theory of character sheaves, Geck and Malle give a more general definition for these matrices [7, 5.1]. For the Ree groups of type  $G_2$ , they obtained the following matrix [7, 5.4]

$$\frac{\sqrt{3}}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & -2 \\ 2 & 2 & 0 & 0 & -2 & 0 \\ 1 & 1 & -1 & -1 & 2 & -2 \\ \sqrt{3} & -\sqrt{3} & -\sqrt{3} & \sqrt{3} & 0 & 0 \end{bmatrix}.$$

We set  $I = \{1, 3, 5, 6, 7, 8, 9, 10\}$ . The Ree group  ${}^2G_2(q)$  has 8 unipotent characters, denoted in [12] by  $\xi_k$  for  $k \in I$ . In [8], Lusztig shows that  $\omega_{\xi_1} = 1$ ,  $\omega_{\xi_3} = 1$ , and

$$\{\omega_{\xi_5}, \omega_{\xi_7}\} = \{\pm i\} \quad \text{and} \quad \{\omega_{\xi_9}, \omega_{\xi_{10}}\} = \left\{ \frac{\pm i - \sqrt{3}}{2} \right\},$$

where  $i \in \overline{\mathbb{Q}_\ell}$  is a root of  $-1$ . This work is completed in [7] by Geck-Malle who proved that  $\{\omega_{\xi_6}, \omega_{\xi_8}\} = \{\pm i\}$ .

The aim of this note is to compute the roots  $\omega_{\xi_k}$  for  $k \in I$ . Moreover, we will also show that the Fourier matrices of the Ree groups of type  $G_2$  satisfy a conjecture of Digne-Michel [5] that we recall in §5. These are new results, which complete works of Lusztig [8] and of Geck-Malle [7] for the Ree groups of type  $G_2$ .

The paper is organized as follows. In Section 2 we fix some notation and we give preliminary results. In Section 3 we give results on the Shintani descents from  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  to  $\mathbf{G}^F$  that we need in order to apply the same method as in [1]. In Section 4 we compute the roots of unity associated as above to the unipotent characters of the Ree groups. Finally, in Section 5 we show that the Fourier matrices for the Ree groups defined by Malle and Geck satisfy the Digne-Michel conjecture.

## 2. NOTATION AND PRELIMINARY RESULTS

**2.1. Notation.** Let  $\mathbf{G}$  be a simple algebraic group of type  $G_2$  over an algebraic closure  $\overline{\mathbb{F}_3}$  of the finite field  $\mathbb{F}_3$  with 3 elements. We denote by  $\Sigma$  the root system of type  $G_2$ , and by  $\Pi = \{a, b\}$  a fundamental system of roots. We choose  $a$  for the short root, and  $b$  for the long one. We denote by  $\Sigma^+ = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$  the set of positive roots with respect to  $\Pi$ . For  $r \in \Sigma$  and  $t \in \overline{\mathbb{F}_3}$ , there is the corresponding Chevalley element  $x_r(t) \in \mathbf{G}$ . We recall that  $\mathbf{G} = \langle x_r(t) \mid r \in \Sigma, t \in \overline{\mathbb{F}_3}^\times \rangle$ . We set

$$\mathbf{U} = \langle x_r(t) \mid r \in \Sigma^+, t \in \overline{\mathbb{F}_3} \rangle \quad \text{and} \quad \mathbf{T} = \langle h_r(t) \mid r \in \Sigma^+, t \in \overline{\mathbb{F}_3} \rangle,$$

where  $h_r(t) = x_{-r}(t^{-1} - 1)x_r(1)x_{-r}(t - 1)x_r(-t^{-1})$ . The subgroup  $\mathbf{T}$  is a maximal torus of  $\mathbf{G}$ , contained in the Borel subgroup  $\mathbf{B} = \mathbf{T}\mathbf{U}$  of  $\mathbf{G}$ . The Weyl group of  $\mathbf{G}$  is  $W = \mathbf{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ .

For every positive integer  $m$ , the Frobenius map  $F_m$  on  $\mathbf{G}$  is defined on the Chevalley generators by setting  $F_m(x_r(t)) = x_r(t^{3^m})$ . As in [3, §12.4], we define an automorphism  $\alpha$  of  $\mathbf{G}$  by setting  $\alpha(x_r(t)) = x_{\rho(r)}(t)$  if  $r$  is a long root and  $\alpha(x_r(t)) = x_{\rho(r)}(t^3)$  if  $r$  is short, where  $\rho$  is the unique angle-preserving, and length-changing bijection of  $\Sigma$  which preserves  $\Pi$ .

Throughout this paper, we fix a positive integer  $n$ . We set  $\theta = 3^n$  and  $q = 3\theta^2$ . We write  $F = \alpha \circ F_n$ . We then have  $F^2 = F_{2n+1}$ . The fixed-point subgroups  $\mathbf{G}^F$  and  $\mathbf{G}^{F^2}$  are the Ree group  ${}^2G_2(q)$  and the finite Chevalley group  $G_2(q)$  respectively. The subgroups  $\mathbf{T}$ ,  $\mathbf{U}$  and  $\mathbf{B}$  are  $F$ -stable. We notice that the automorphism of  $W$  induced by  $F$  has order 2.

Moreover, the Chevalley relations are, for  $u, v \in \overline{\mathbb{F}}_3$ :

$$\begin{aligned} x_a(t)x_b(u) &= x_b(u)x_a(t)x_{a+b}(-tu)x_{3a+b}(t^3u)x_{2a+b}(-t^2u)x_{3a+2b}(t^3u^2) \\ x_a(t)x_{a+b}(u) &= x_{a+b}(u)x_a(t)x_{2a+b}(tu) \\ x_b(t)x_{3a+b}(u) &= x_{3a+b}(u)x_b(t)x_{3a+2b}(tu) \\ x_{a+b}(t)x_{3a+b}(u) &= x_{3a+b}(u)x_{a+b}(t) \\ x_{a+b}(t)x_{2a+b}(u) &= x_{2a+b}(u)x_{a+b}(t) \\ x_{a+b}(t)x_{3a+2b}(u) &= x_{3a+2b}(u)x_{a+b}(t) \\ x_{2a+b}(t)x_{3a+b}(u) &= x_{3a+b}(u)x_{2a+b}(t) \\ x_{2a+b}(t)x_{3a+2b}(u) &= x_{3a+2b}(u)x_{2a+b}(t) \end{aligned}$$

We fix a root  $\alpha_0 \in \overline{\mathbb{F}}_3^\times$  of  $X^q - X + 1$ , and we set  $\xi = \alpha_0^3 - \alpha_0$ . We have

$$\xi^q = \alpha_0^{3q} - \alpha_0^q = \alpha_0^3 - 1 - \alpha_0 + 1 = \xi.$$

Therefore  $\xi \in \mathbb{F}_q$ . Moreover,  $X^3 - X - \xi \in \mathbb{F}_q[X]$  is irreducible over  $\mathbb{F}_q$ . Otherwise there is a  $t \in \mathbb{F}_q$  with  $t^3 - t - \xi = 0$ , implying  $(t - \alpha_0)^3 = (t - \alpha_0)$ . However,  $t \neq \alpha_0$  (because  $\alpha_0 \notin \mathbb{F}_q$ ). Thus  $(t - \alpha_0)^2 = 1$ . It follows that  $\alpha_0 = t \pm 1 \in \mathbb{F}_q$ . This is a contradiction.

The character table of  $\mathbf{G}^{F^2}$  was computed by Enomoto in [6]. The description of this table depends on an element  $\xi_0 \in \mathbb{F}_q$  such that  $X^3 - X - \xi_0$  is an irreducible polynomial over  $\mathbb{F}_q$ . In the following we choose  $\xi_0 = \xi$ .

We recall that the unipotent regular class  $u_{\text{reg}}$  of  $\mathbf{G}$  splits in 3 classes  $A_{51}$ ,  $A_{52}$  and  $A_{53}$  in  $\mathbf{G}^{F^2}$ , with representatives  $x_a(1)x_b(1)$ ,  $x_a(1)x_b(1)x_{3a+b}(\xi)$  and  $x_a(1)x_b(1)x_{3a+b}(-\xi)$  respectively. Moreover,  $u_{\text{reg}}$  also splits in 3 classes in  $\mathbf{G}^F$  whose representatives are not conjugate in  $\mathbf{G}^{F^2}$ . We denote by  $Y_1$ ,  $Y_2$  and  $Y_3$  representatives with  $Y_1 \in A_{51}$ ,  $Y_2 \in A_{52}$  and  $Y_3 \in A_{53}$ .

The group  $\mathbf{G}^F$  has  $q + 8$  conjugacy classes. More precisely, we recall that the system of representatives of classes of  $\mathbf{G}^F$  given in [2, 4.1] is described as follows.

- The trivial element 1.
- The element  $J = h_{a+b}(-1)$  which has a centralizer of order  $q(q-1)(q+1)$ .
- The element  $X = x_{2a+b}(1)x_{3a+2b}(1)$  which has centralizer order  $q^3$ .
- The elements  $T = x_{a+b}(1)x_{3a+b}(1)$  and  $T^{-1}$  whose centralizers have order  $2q^2$ .
- The elements  $Y_1$ ,  $Y_2$  and  $Y_3$  described as above whose centralizers have order  $3q$ .
- The elements  $TJ$  and  $T^{-1}J$  whose centralizers have order  $2q$ .

- A family  $R$  of  $(q-3)/2$  semisimple regular elements with centralizer order  $q-1$ .
- A family  $S$  of  $(q-3)/6$  semisimple regular elements with centralizer order  $q+1$ .
- A family  $V$  of  $(q-3\theta)/6$  semisimple regular elements with centralizer order  $q-3\theta+1$ .
- A family  $M$  of  $(q+3\theta)/2$  semisimple regular elements with centralizer order  $q+3\theta+1$ .

In [2, 4.5] we give the class fusion between  $\mathbf{G}^F$  and  $\mathbf{G}^{F^2}$ . The character table of  $\mathbf{G}^F$  with respect to this parametrization is given in [2, 7.2]. We notice that there are some misprints in [2, 7.2] for the values of  $\xi_9$  and  $\xi_{10}$  on  $Y_2$  and  $Y_3$ . Indeed we have

$$\xi_9(Y_2) = \xi_{10}(Y_3) = \theta(1 + i\sqrt{3})/2 \quad \text{and} \quad \xi_9(Y_3) = \xi_{10}(Y_2) = \theta(1 - i\sqrt{3})/2.$$

A system of representatives of the conjugacy classes of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  is computed in [2, 4.2]. It is shown that the following elements are representatives of the outer classes of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  (i.e. the classes of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  lying in the coset  $\mathbf{G}^{F^2}.F$ ):

- The element  $F$ , which has centralizer order  $2q^3(q^2-1)q^2-q+1$ .
- The element  $h_0.F$  with  $h_0 = h_a(-1)$ , which has centralizer order  $2q(q-1)(q+1)$ .
- The element  $X.F$ , with centralizer order  $2q^3$ .
- The elements  $T.F$  and  $T^{-1}.F$ , whose centralizers have order  $6q^2$ .
- The elements  $Y_1.F$ ,  $Y_2.F$  and  $Y_3.F$ , whose centralizers have order  $6q$ .
- The elements  $\eta h_0.F$  and  $\eta^{-1}h_0.F$  with  $\eta = x_{a+b}(1)x_{3a+b}(-1)$ , whose centralizers have order  $2q$ .
- A family  $R'$  of  $(q-3)/2$  elements with centralizer order  $q-1$ .
- A family  $S'$  of  $(q-3)/6$  elements with centralizer order  $q+1$ .
- A family  $V'$  of  $(q-3\theta)/6$  elements with centralizer order  $q-3\theta+1$ .
- A family  $M'$  of  $(q+3\theta)/2$  elements with centralizer order  $q+3\theta+1$ .

Finally, we recall that the character table of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  is computed in [2, 1.1].

**2.2. Some results.** We will need the following results.

**Lemma 2.1.** *We set  $E = \{t^3 - t \mid t \in \mathbb{F}_q\}$ . Then every element  $x \in \mathbb{F}_q$  can be written uniquely as  $x = \eta_x \xi + y_x$  with  $y_x \in E$  and  $\eta_x \in \mathbb{F}_3$ . Moreover let  $u \in \mathbf{U}^{F^2}$  be such that*

$$u = x_a(1)x_b(1)x_{a+b}(t_{a+b})x_{3a+b}(t_{3a+b})x_{2a+b}(t_{2a+b})x_{3a+2b}(t_{3a+2b}),$$

*for  $t_{a+b}, t_{3a+b}, t_{2a+b}, t_{3a+2b} \in \mathbb{F}_q$ . For  $u$  as above, we set  $p(u) = t_{a+b} + t_{3a+b}$ . Then  $u \in A_{51}$  if  $\eta_{p(u)} = 0$ ,  $u \in A_{52}$  if  $\eta_{p(u)} = 1$ , and  $u \in A_{53}$  if  $\eta_{p(u)} = -1$ .*

*Proof.* This is a consequence of the Chevalley relations.  $\square$

We remark that the map  $\mathbb{F}_q \rightarrow \mathbb{F}_3$ ,  $x \mapsto \eta_x$  is an additive morphism. We now describe the elements  $Y_1$ ,  $Y_2$  and  $Y_3$  more precisely.

**Lemma 2.2.** *For  $u = \pm \xi$ , we set*

$$\alpha(1) = x_a(1)x_b(1)x_{a+b}(1)x_{2a+b}(1) \quad \text{and} \quad \beta(u) = x_{a+b}(u^\theta)x_{3a+b}(u).$$

*As previously, we denote by  $\eta_1 \in \mathbb{F}_3$  the unique element such that  $1 = \eta_1 \xi + t^3 - t$  for some  $t \in \mathbb{F}_q$ .*

- If  $n \equiv 1 \pmod{3}$ , then  $\eta_1 = 0$ . We choose  $Y_1 = \alpha(1)$ ,  $Y_2 = \alpha(1)\beta(-\xi)$  and  $Y_3 = \alpha(1)\beta(\xi)$ .
- If  $n \equiv 0 \pmod{3}$ , then  $\eta_1 = -1$ . We choose  $Y_1 = \alpha(1)\beta(-\xi)$ ,  $Y_2 = \alpha(1)\beta(\xi)$  and  $Y_3 = \alpha(1)$ .
- If  $n \equiv -1 \pmod{3}$ , then  $\eta_1 = 1$ . We choose  $Y_1 = \alpha(1)\beta(\xi)$ ,  $Y_2 = \alpha(1)$  and  $Y_3 = \alpha(1)\beta(-\xi)$ .

*Proof.* We have

$$p(\alpha(1)) = 1, \quad p(\alpha(1)\beta(\xi)) = 1 + \xi + \xi^\theta \quad \text{and} \quad p(\alpha(1)\beta(-\xi)) = 1 - \xi - \xi^\theta.$$

We discuss  $\eta_1$ . There is an element  $t \in \mathbb{F}_q$  such that

$$1 = \eta_1 \xi + t^3 - t.$$

For  $0 \leq j \leq 2n$ , we take the  $3^j$ -power of the last relation, and sum all new obtained relations. Thus we obtain

$$2n + 1 = \eta_1(\xi + \xi^3 + \cdots + \xi^{3^{2n}}).$$

However,  $\xi = \alpha_0^3 - \alpha_0$  with  $\alpha_0^q = \alpha_0 - 1$ . Hence  $\xi + \cdots + \xi^{3^{2n}} = \alpha_0^q - \alpha_0 = -1$ . It follows that

$$2n + 1 + \eta_1 \equiv 0 \pmod{3}.$$

Moreover, we remark that  $\xi^\theta = (\xi + \cdots + \xi^{\theta/3})^3 - (\xi + \cdots + \xi^{\theta/3}) + \xi$ . We deduce that  $\eta_{\xi^\theta} = 1$ .

- If  $n \equiv 1 \pmod{3}$ , then  $\eta_1 = 0$ . Therefore  $\eta_{1+\xi+\xi^\theta} = -1$  and  $\eta_{1-\xi-\xi^\theta} = 1$ .
- If  $n \equiv 0 \pmod{3}$ , then  $\eta_1 = -1$ . Therefore  $\eta_{1+\xi+\xi^\theta} = 1$  and  $\eta_{1-\xi-\xi^\theta} = 0$ .
- If  $n \equiv -1 \pmod{3}$ , then  $\eta_1 = 1$ . Therefore  $\eta_{1+\xi+\xi^\theta} = 0$  and  $\eta_{1-\xi-\xi^\theta} = -1$ .

The result follows.  $\square$

**Lemma 2.3.** *Let  $\alpha$  and  $\beta$  be two elements of  $\mathbb{F}_q$ . We consider the following system of equations (S)*

$$\begin{cases} y^\theta - x & = 1 \\ x^{3\theta} - y & = 1 \\ t^\theta - z + 1 + x^{3\theta+1} & = \alpha \\ z^{3\theta} - t - 1 - x^{3\theta+3} & = \beta \end{cases}$$

*If  $x_0$  is a root in  $\overline{\mathbb{F}_3}$  of  $X^q - X + 1$  and  $t_0$  is a root in  $\overline{\mathbb{F}_3}$  of  $X^q - X - x_0^{3\theta} - \beta - \alpha^{3\theta}$ , then the tuple  $(x_0, x_0^{3\theta} - 1, t_0, x_0^{3\theta+1} + t_0^\theta + 1 - \alpha)$  is a solution of (S) in  $\overline{\mathbb{F}_3}$ .*

*Proof.* If  $y_0 = x_0^{3\theta} - 1$ , then  $y_0^\theta = x_0^q - 1 = x_0 + 1$ . If  $z_0 = x_0^{3\theta+1} + t_0^\theta + 1 - \alpha$ , then

$$\begin{aligned} z_0^{3\theta} &= x_0^{3q} x_0^{3\theta} + t_0^q + 1 - \alpha^{3\theta} \\ &= (x_0^3 - 1)x_0^{3\theta} + t_0 + x_0^{3\theta} + \beta + \alpha^{3\theta} + 1 - \alpha^{3\theta} \\ &= x_0^{3\theta+3} + t_0 + \beta + 1 \end{aligned}$$

The result follows.  $\square$

**Remark 2.4.** *For the solution of the system (S) given in Lemma 2.3, we remark that we can choose  $x_0$  independently of  $\alpha$  and  $\beta$ .*

## 3. SHINTANI DESCENTS

**3.1. Definition.** A reference for this section is for example [5]. We keep the notation as above. We denote by  $L_F : \mathbf{G} \rightarrow \mathbf{G}$ ,  $x \mapsto x^{-1}F(x)$  the Lang map associated to  $F$ . Since  $\mathbf{G}$  is a simple algebraic group, it follows that  $\mathbf{G}$  is connected. Hence  $L_F$  and  $L_{F^2}$  are surjective. Moreover,  $L_F(x) \in \mathbf{G}^{F^2}$  for some  $x \in \mathbf{G}$  if and only if  $L_{F^2}(x^{-1}) \in \mathbf{G}^F$ , and the correspondence

$$L_{F^2}(x^{-1}) \in \mathbf{G}^F \longleftrightarrow L_F(x).F \in \mathbf{G}^{F^2} \rtimes \langle F \rangle \quad \text{for } x \in \mathbf{G},$$

induces a bijection from the outer classes of the group  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  to the classes of  $\mathbf{G}^F$ . We denote this correspondence by  $N_{F/F^2}$ . Furthermore we have [5, §I.7]

$$(1) \quad |\mathbf{C}_{\mathbf{G}^{F^2} \rtimes \langle F \rangle}(L_F(x).F)| = 2|\mathbf{C}_{\mathbf{G}^F}(L_{F^2}(x^{-1}))| \quad \text{for } x \in \mathbf{G}.$$

For every class function  $\psi$  on  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$ , we define the Shintani  $\text{Sh}_{F^2/F}(\psi)$  of  $\psi$  by  $\text{Sh}_{F^2/F}(\psi) = \psi \circ N_{F/F^2}$ .

**3.2. Shintani correspondence from  $\mathbf{G}^F$  to  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  in type  $G_2$ .**

**Lemma 3.1.** *We keep the notation as above. We set  $T = \beta(1)$ . We have*

$$N_{F/F^2}([T]_{\mathbf{G}^F}) = [T.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \quad \text{and} \quad N_{F/F^2}([T^{-1}]_{\mathbf{G}^F}) = [T^{-1}.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle}.$$

Here  $[x]_G$  denotes the conjugacy class of  $x$  in  $G$ .

*Proof.* We set  $x = x_{a+b}(\alpha_0)x_{3a+b}(\alpha_0^{3\theta} - 1)$ . Then we have

$$\begin{aligned} L_F(x) &= x_{a+b}((\alpha_0^{3\theta} - 1)^\theta - \alpha_0)x_{3a+b}(\alpha_0^{3\theta} - (\alpha_0^{3\theta} - 1)) \\ &= x_{a+b}(\alpha_0 - 2 - \alpha_0)x_{3a+b}(1) \\ &= \beta(1) \end{aligned}$$

and  $L_{F^2}(x^{-1}) = x_{a+b}(\alpha_0 - \alpha_0^q)x_{3a+b}(\alpha_0 - \alpha_0^q) = \beta(1)$ . □

**Lemma 3.2.** *We keep the notation as above. Then we have*

$$\begin{aligned} N_{F/F^2}([Y_1]_{\mathbf{G}^F}) &= [Y_3.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\ N_{F/F^2}([Y_2]_{\mathbf{G}^F}) &= [Y_1.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\ N_{F/F^2}([Y_3]_{\mathbf{G}^F}) &= [Y_2.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle}. \end{aligned}$$

*Proof.* Let  $u = x_a(1)x_b(1)x_{a+b}(\alpha)x_{3a+b}(\beta)y \in \mathbf{U}^{F^2}$  with  $y \in \mathbf{Z}(\mathbf{U}^{F^2})$ . Since  $\mathbf{U}$  is connected, there is  $x \in \mathbf{U}$  such that  $L_F(x) = u$ . Then there are  $z \in \mathbf{Z}(\mathbf{U}^{F^2})$  and  $t_a, t_b, t_{a+b}$  and  $t_{3a+b}$  in  $\overline{\mathbb{F}}_3$  such that

$$x = x_a(t_a)x_b(t_b)x_{a+b}(t_{a+b})x_{3a+b}(t_{3a+b})z.$$

Using the Chevalley relations, we have

$$\begin{aligned} L_F(x) &= x_a(t_b^\theta - t_a)x_b(t_a^{3\theta} - t_b)x_{a+b}(t_at_b - t_{a+b} + t_b^\theta t_a^{3\theta} + t_{3a+b}^\theta - t_b^{\theta+1}) \\ &\quad x_{3a+b}(t_{a+b}^{3\theta} - t_b^{3\theta} t_a^{3\theta} - t_a^3 t_b - t_{3a+b} + t_b^{3\theta+1})z', \end{aligned}$$

for some  $z' \in \mathbf{Z}(\mathbf{U}^{F^2})$ . However, using the uniqueness of the Chevalley decomposition we have the system of equations  $(S')$

$$\begin{cases} t_b^\theta - t_a &= 1 \\ t_a^{3\theta} - t_b &= 1 \\ t_at_b - t_{a+b} + t_b^\theta t_a^{3\theta} + t_{3a+b}^\theta - t_b^{\theta+1} &= \alpha \\ t_{a+b}^{3\theta} - t_b^{3\theta} t_a^{3\theta} - t_a^3 t_b - t_{3a+b} + t_b^{3\theta+1} &= \beta \end{cases}$$

We deduce from the two first equations that  $t_a^q - t_a = -1$  and  $t_b^q - t_b = -1$ .

Moreover, there is  $z'' \in \mathbf{Z}(\mathbf{U}^{F^2})$  such that

$$\begin{aligned} L_{F^2}(x^{-1}) &= x_a(t_a - t_a^q)x_b(t_b - t_b^q)x_{a+b}(t_a^q(t_b^q - t_b) + t_{a+b} - t_{a+b}^q) \\ &\quad x_{3a+b}(t_{3a+b} - t_{3a+b}^q - t_a^{3q}(t_b - t_b^q))z'' \\ &= x_a(1)x_b(1)x_{a+b}(t_{a+b} - t_{a+b}^q - t_a^q)x_{3a+b}(t_{3a+b} - t_{3a+b}^q + t_a^{3q})z''. \end{aligned}$$

Hence using Lemma 2.2 in order to find the  $\mathbf{G}^F$ -class of  $L_{F^2}(x^{-1}) \in \mathbf{G}^F$ , it is sufficient to find the  $\mathbf{G}^{F^2}$ -class of  $L_{F^2}(x^{-1})$ . However using Lemma 2.1, we have to evaluate  $\eta_{p(L_{F^2}(x^{-1}))}$  where  $p$  is defined as in Lemma 2.1. We have

$$\begin{aligned} p(L_{F^2}(x^{-1})) &= t_{3a+b} - t_{3a+b}^q + t_a^{3q} + t_{a+b} - t_{a+b}^q - t_a^q \\ &= t_{3a+b} - t_{3a+b}^q + t_{a+b} - t_{a+b}^q + t_a^3 - t_a. \end{aligned}$$

Using the equations of the system  $(S')$ , we deduce that

$$\begin{cases} t_a^{3\theta+1} + t_{3a+b}^\theta - t_{a+b} + 1 &= \alpha \\ -t_a^{3\theta+3} + t_{a+b}^{3\theta} - t_{3a+b} - 1 &= \beta \end{cases}$$

It follows that

$$\begin{cases} t_a^{(3\theta+1)3\theta} - t_a^{3\theta+3} + t_{3a+b}^q - t_{3a+b} &= \alpha^{3\theta} + \beta \\ t_a^{3\theta+1} - t_a^{(3\theta+3)\theta} + t_{a+b}^q - t_{a+b} &= \alpha + \beta^\theta \end{cases}$$

Adding these two equations, we obtain

$$t_{a+b}^q - t_{a+b} + t_{3a+b}^q - t_{3a+b} = \alpha + \alpha^{3\theta} + \beta + \beta^\theta.$$

Hence we have

$$p(L_{F^2}(x^{-1})) = -(\alpha + \alpha^{3\theta} + \beta + \beta^\theta) + t_a^3 - t_a.$$

Moreover, we remark that the system  $(S')$  is equivalent to the system  $(S)$ . We use for  $(t_a, t_b, t_{a+b}, t_{3a+b})$  the solution described in Lemma 2.3, choosing  $t_a = \alpha_0$ , which is possible as we have seen in Remark 2.4. Thus

$$p(L_{F^2}(x^{-1})) = -(\alpha + \alpha^{3\theta} + \beta + \beta^\theta) + \xi.$$

We suppose now that  $n \equiv 1 \pmod{3}$ . Using Lemma 2.2, we have

$$\begin{aligned} Y_1 &= x_a(1)x_b(1)x_{a+b}(1)x_{2a+b}(1) \\ Y_2 &= x_a(1)x_b(1)x_{a+b}(1 - \xi^\theta)x_{3a+b}(-\xi)x_{2a+b}(1) \\ Y_3 &= x_a(1)x_b(1)x_{a+b}(1 + \xi^\theta)x_{3a+b}(\xi)x_{2a+b}(1) \end{aligned}$$

For  $(\alpha, \beta) = (1, 0)$ ,  $(\alpha, \beta) = (1 - \xi^\theta, -\xi)$  and  $(\alpha, \beta) = (1 + \xi^\theta, \xi)$  respectively, we deduce that  $\eta_{p(L_{F^2}(x^{-1}))}$  is equal to 1, -1, and 0 respectively, because  $\eta_1 = 0$ . Thus

$$\begin{aligned} N_{F/F^2}([Y_1]_{\mathbf{G}^F}) &= [Y_3 \cdot F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\ N_{F/F^2}([Y_2]_{\mathbf{G}^F}) &= [Y_1 \cdot F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\ N_{F/F^2}([Y_3]_{\mathbf{G}^F}) &= [Y_2 \cdot F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \end{aligned}$$

We proceed similarly when  $n \equiv 0 \pmod{3}$  and  $n \equiv -1 \pmod{3}$ . The result follows.  $\square$

**Proposition 3.3.** *We keep the same notation as above. Then we have*

$$\begin{aligned}
N_{F/F^2}([1]_{\mathbf{G}^F}) &= [F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([X]_{\mathbf{G}^F}) &= [X.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([J]_{\mathbf{G}^F}) &= [h_0.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([T]_{\mathbf{G}^F}) &= [T.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([T^{-1}]_{\mathbf{G}^F}) &= [T^{-1}.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([Y_1]_{\mathbf{G}^F}) &= [Y_3.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([Y_2]_{\mathbf{G}^F}) &= [Y_1.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
N_{F/F^2}([Y_3]_{\mathbf{G}^F}) &= [Y_2.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \\
\{N_{F/F^2}([JT]_{\mathbf{G}^F}), N_{F/F^2}([JT^{-1}]_{\mathbf{G}^F})\} &= \{[\eta h_0.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle}, [\eta^{-1} h_0.F]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle}\} \\
\{N_{F/F^2}([t]_{\mathbf{G}^F}) \mid t \in R\} &= \{[t]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \mid t \in R'\} \\
\{N_{F/F^2}([t]_{\mathbf{G}^F}) \mid t \in S\} &= \{[t]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \mid t \in S'\} \\
\{N_{F/F^2}([t]_{\mathbf{G}^F}) \mid t \in V\} &= \{[t]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \mid t \in V'\} \\
\{N_{F/F^2}([t]_{\mathbf{G}^F}) \mid t \in M\} &= \{[t]_{\mathbf{G}^{F^2} \rtimes \langle F \rangle} \mid t \in M'\}
\end{aligned}$$

*Proof.* To prove this result, we essentially use Relation (1) comparing the orders of centralizers of the representatives of classes of  $\mathbf{G}^F$ , and of the representatives of the outer classes of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  recalled in §2. Moreover, for the classes  $[T]_{\mathbf{G}^F}$ ,  $[T^{-1}]_{\mathbf{G}^F}$ , and  $[Y_1]_{\mathbf{G}^F}$ ,  $[Y_2]_{\mathbf{G}^F}$ ,  $[Y_3]_{\mathbf{G}^F}$ , we use Lemma 3.1 and Lemma 3.2 respectively.  $\square$

**3.3. Shintani descents of the unipotent characters in type  $G_2$ .** The  $F$ -stable unipotent characters of  $\mathbf{G}^{F^2}$  are denoted by  $1_{\mathbf{G}^{F^2}}$ ,  $\theta_1$ ,  $\theta_2$ ,  $\theta_5$ ,  $\theta_{10}$ ,  $\theta_{11}$ ,  $\theta_{12}[1]$  and  $\theta_{12}[-1]$  in [6]. Their degrees are  $1$ ,  $\frac{1}{6}q(q+1)^2(q^2+q+1)$ ,  $\frac{1}{2}q(q+1)(q^3+1)$ ,  $q^6$ ,  $\frac{1}{6}q(q-1)^2(q^2-q+1)$ ,  $\frac{1}{2}q(q-1)(q^3-1)$ ,  $\frac{1}{3}q(q^2-1)^2$  and  $\frac{1}{3}q(q^2-1)^2$  respectively. These characters extends to  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$ . Let  $\chi$  be such a character. If  $\chi$  has an extension such that its value on  $F$  is not zero, then we denote by  $\tilde{\chi}$  the extension of  $\chi$  such that  $\tilde{\chi}(F) > 0$ . In [2, 5.6] we have seen that this is always the case except for  $\theta_2$  and  $\theta_{10}$ . Then we choose for  $\tilde{\theta}_2$  and  $\tilde{\theta}_{10}$  the extensions such that

$$\tilde{\theta}_2(\eta h_0.F) = \sqrt{q} \quad \text{and} \quad \tilde{\theta}_{10}(T.F) = \theta^2 \sqrt{3}i.$$

Moreover, there is a misprint in [2, 5.6]. Indeed, we have  $\tilde{\theta}_{10}(Y_2) = -\theta\sqrt{3}i$  and  $\tilde{\theta}_{10}(Y_3) = \theta\sqrt{3}i$ .

**Proposition 3.4.** *We keep the same notation as above. Then we have*

$$\begin{aligned}
\text{Sh}_{F^2/F}(1_{\mathbf{G}^{F^2} \rtimes \langle F \rangle}) &= 1_{\mathbf{G}^F} \\
\text{Sh}_{F^2/F}(\tilde{\theta}_1) &= \frac{\sqrt{3}}{6} (i\xi_5 + i\xi_6 - i\xi_7 - i\xi_8 + (\sqrt{3} - i)\xi_9 + (\sqrt{3} + i)\xi_{10}) \\
\text{Sh}_{F^2/F}(\tilde{\theta}_2) &= \pm \frac{1}{2} (-i\xi_5 + i\xi_6 + i\xi_7 - i\xi_8) \\
\text{Sh}_{F^2/F}(\tilde{\theta}_5) &= \xi_3 \\
\text{Sh}_{F^2/F}(\tilde{\theta}_{10}) &= \frac{\sqrt{3}}{6} (i\xi_5 + i\xi_6 + i\xi_7 + i\xi_8 + (\sqrt{3} - i)\xi_9 - (i + \sqrt{3})\xi_{10}) \\
\text{Sh}_{F^2/F}(\tilde{\theta}_{11}) &= \frac{1}{2} (\xi_5 - \xi_6 + \xi_7 - \xi_8) \\
\text{Sh}_{F^2/F}(\tilde{\theta}_{12}[1]) &= \frac{\sqrt{3}}{6} ((\sqrt{3} - i)\xi_5 + (\sqrt{3} - i)\xi_6 + (\sqrt{3} + i)\xi_9) \\
\text{Sh}_{F^2/F}(\tilde{\theta}_{12}[-1]) &= \frac{\sqrt{3}}{6} ((\sqrt{3} + i)\xi_7 + (\sqrt{3} + i)\xi_8 + (\sqrt{3} - i)\xi_{10})
\end{aligned}$$



*Proof.* We use Proposition 3.3 and [2, 5.6] to obtain the values of the Shintani descents as class functions on  $\mathbf{G}^F$ . Using the character table of  $\mathbf{G}^F$  in [12], we decompose them in the basis of irreducible characters of  $\mathbf{G}^F$ . The result follows.  $\square$

#### 4. EIGENVALUES OF THE FROBENIUS FOR THE REE GROUPS OF TYPE $G_2$

As an application of Proposition 3.4 we compute in this section the roots of unity associated to the unipotent characters of the Ree groups of type  $G_2$ , as explained in Section 1.

**Theorem 4.1.** *We keep the same notation as above. The roots of unity associated to the unipotent characters of  ${}^2G_2(q)$  are*

$\chi$	$\xi_1$	$\xi_3$	$\xi_5$	$\xi_6$	$\xi_7$	$\xi_8$	$\xi_9$	$\xi_{10}$
$\omega_\chi$	1	1	$-i$	$-i$	$i$	$i$	$\frac{-\sqrt{3}+i}{2}$	$\frac{-\sqrt{3}-i}{2}$

*Proof.* We first recall a result of Digne-Michel. Let  $\rho$  be an irreducible character of  $W$ . Using the Harish-Chandra theory, we can associate to  $\rho$  an irreducible character  $\chi_\rho$  of the principal series of  $\mathbf{G}^{F^2}$ , that is the set of irreducible constituents of

$$\Phi = \text{Ind}_{\mathbf{B}^{F^2}}^{\mathbf{G}^{F^2}}(1_{\mathbf{B}^{F^2}}).$$

The group  $\langle F \rangle$  acts on  $\text{Irr}(W)$ . If  $\rho$  is  $F$ -stable, then  $\rho$  extends to  $W \rtimes \langle F \rangle$ , and has exactly 2 extensions denoted by  $\tilde{\rho}$  and  $\varepsilon\tilde{\rho}$ , where  $\varepsilon$  is the non-trivial character of  $W \rtimes \langle F \rangle$  which has  $W$  as its kernel. However Malle shows in [11, 1.5] that the irreducible characters of  $W \rtimes \langle F \rangle$  are in 1-1 correspondence with the constituents of  $\tilde{\Phi} = \text{Ind}_{\mathbf{G}^{F^2}}^{\mathbf{G}^{F^2} \rtimes \langle F \rangle}(\Phi)$ . In particular, if  $\rho$  is  $F$ -stable, then so is  $\chi_\rho$ . Hence, the characters  $\chi_{\tilde{\rho}}$  and  $\chi_{\varepsilon\tilde{\rho}}$  of  $\tilde{\Phi}$  corresponding to  $\tilde{\rho}$  and  $\varepsilon\tilde{\rho}$  respectively, are the two extensions of  $\chi_\rho$  to  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$ . Moreover, we recall that the almost character of  $\mathbf{G}^F$  corresponding to  $\tilde{\rho}$  is defined by

$$\mathcal{R}_{\tilde{\rho}} = \sum_{w \in W} \tilde{\rho}(w.F) R_w.$$

The main theorem [5, 2.3] asserts that

$$(2) \quad \text{Sh}_{F^2/F}(\chi_{\tilde{\rho}}) = \sum_{V \in \mathcal{U}(\mathbf{G}^F)} \langle \mathcal{R}_{\tilde{\rho}}, V \rangle_{\mathbf{G}^F} \omega_V V.$$

Furthermore the almost characters of the Ree groups are computed by Geck and Malle in [7, 2.2]. More precisely, the  $F$ -stable characters of  $W$  are  $1_W$ ,  $\text{sgn}$ , and the two characters of degree 2 of  $W$ , denoted by  $2_1$  and  $2_2$ . The character  $2_1$  is chosen such that it takes the value  $-2$  on the Coxeter element of  $W$ . Then for the extensions of these characters chosen in [7], we have

$$\begin{aligned} \mathcal{R}_{1_{W \rtimes \langle F \rangle}} &= 1_{\mathbf{G}^F} \\ \mathcal{R}_{\widetilde{\text{sgn}}} &= \xi_3 \\ \mathcal{R}_{\tilde{2}_1} &= \frac{\sqrt{3}}{6}(\xi_5 + \xi_6 + \xi_7 + \xi_8 + 2\xi_9 + 2\xi_{10}) \\ \mathcal{R}_{\tilde{2}_2} &= \frac{1}{2}(\xi_5 - \xi_6 + \xi_7 - \xi_8) \end{aligned}$$

Moreover, using [4, p.112,p.150], we deduce that

$$\text{Ind}_{\mathbf{B}^{F^2}}^{\mathbf{G}^{F^2}}(1_{\mathbf{B}^F}) = 1_{\mathbf{G}^{F^2}} + \theta_5 + \theta_3 + \theta_4 + 2\theta_1 + 2\theta_2,$$

and  $1_{\mathbf{G}^{F^2}} = \chi_{1_W}$ ,  $\theta_5 = \chi_{\text{sgn}}$ ,  $\theta_1 = \chi_{2_1}$  and  $\theta_2 = \chi_{2_2}$ . Hence we have

$$\chi_{\tilde{2}_1} \in \{\tilde{\theta}_1, \varepsilon \tilde{\theta}_1\},$$

where  $\varepsilon$  denotes now the non trivial character of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  containing  $\mathbf{G}^{F^2}$  in its kernel. Therefore, using Relation (2) we deduce that

$$\pm \text{Sh}_{F^2/F}(\tilde{\theta}_1) = \frac{\sqrt{3}}{6} (\omega_{\xi_5} \xi_5 + \omega_{\xi_6} \xi_6 + \omega_{\xi_7} \xi_7 + \omega_{\xi_8} \xi_8 + 2\omega_{\xi_9} \xi_9 + 2\omega_{\xi_{10}} \xi_{10}).$$

Hence we deduce using Proposition 3.4 that

$$\omega_{\xi_9} = \pm \sqrt{3} \langle \text{Sh}_{F^2/F}(\theta_1), \xi_9 \rangle_{\mathbf{G}^F} = \pm \frac{\sqrt{3} - i}{2}.$$

However using the result of Lusztig [8], we know that  $\omega_{\xi_9} = (\pm i - \sqrt{3})/2$ . We then deduce that  $\chi_{\tilde{2}_1} = \varepsilon \tilde{\theta}_1$  and that

$$\omega_{\xi_9} = \frac{-\sqrt{3} + i}{2}.$$

We immediately obtain the other roots using Proposition 3.4.  $\square$

**Remark 4.2.** *To determine the roots of unity of the unipotent characters of the Ree groups of type  $G_2$ , the preceding proof shows that we only need to know the Shintani descent of  $\tilde{\theta}_1$ .*

## 5. CONJECTURE OF DIGNE-MICHEL FOR THE REE GROUPS OF TYPE $G_2$

In [5] Digne and Michel state a conjecture on the decomposition in irreducible constituents of the Shintani descents. We recall this conjecture in the special case that  $F \in \text{Aut}(W)$  has order 2. If  $\chi$  is an  $F$ -stable irreducible character of  $\mathbf{G}^{F^2}$ , and if  $\tilde{\chi}$  denotes an extension of  $\chi$  to  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$ , then it is conjectured that:

- The irreducible constituents of  $\text{Sh}_{F^2/F}(\tilde{\chi})$  are unipotent.
- Up to a normalization, the coefficients of  $\text{Sh}_{F^2/F}$  in the basis  $\text{Irr}(\mathbf{G}^F)$  only depend on the coefficients of the Fourier matrix, and on the roots of unity attached to the unipotent characters of  $\mathbf{G}^F$  as above. More precisely, there is a root of unity  $u$  such that

$$\pm u \text{Sh}_{F^2/F}(\tilde{\chi}) = \sum_{V \in \mathcal{U}(\mathbf{G}^F)} f_V \omega_V V,$$

where  $(f_V, V \in \mathcal{U}(\mathbf{G}^F))$  is, up to a sign, a row of the Fourier matrix.

**Theorem 5.1.** *The conjecture of Digne-Michel on the decomposition of the Shintani descents of unipotent characters holds in type  $G_2$  for the Frobenius map  $F$  that defines the Ree group  ${}^2G_2(q)$ .*

*Proof.* We set  $u_{12[1]} = \frac{1}{2}(-1 + \sqrt{3}i)$  and  $u_{12[-1]} = \frac{1}{2}(1 + \sqrt{3}i)$ . We remark that

$$\begin{aligned} u_{12[1]}(\sqrt{3} - i) &= 2i, & u_{12[1]}(\sqrt{3} + i) &= -\sqrt{3} + i, \\ u_{12[-1]}(\sqrt{3} + i) &= 2i, & u_{12[-1]}(\sqrt{3} - i) &= \sqrt{3} + i \end{aligned}$$

Therefore, using Proposition 3.4 we have

$$\begin{aligned} u_{12[1]} \operatorname{Sh}_{F^2/F}(\tilde{\theta}_{12[1]}) &= \frac{\sqrt{3}}{6} (2i\xi_5 + 2i\xi_6 + (i - \sqrt{3})\xi_9) \\ &= \frac{\sqrt{3}}{6} \left( -2(-i)\xi_5 + -2(-i)\xi_6 + 2\frac{1}{2}(i - \sqrt{3})\xi_9 \right). \end{aligned}$$

Similarly, we obtain

$$u_{12[-1]} \operatorname{Sh}_{F^2/F}(\tilde{\theta}_{12[-1]}) = \frac{\sqrt{3}}{6} \left( 2i\xi_7 + 2i\xi_8 - \frac{1}{2}(-i - \sqrt{3})\xi_{10} \right).$$

Moreover, we have

$$\begin{aligned} \operatorname{Sh}_{F^2/F}(\tilde{\theta}_2) &= \pm \frac{\sqrt{3}}{6} (\sqrt{3}(-i)\xi_5 - \sqrt{3}(-i)\xi_6 + \sqrt{3}i\xi_7 - \sqrt{3}i\xi_8), \\ \operatorname{Sh}_{F^2/F}(\tilde{\theta}_{10}) &= \frac{\sqrt{3}}{6} \left( -(-i)\xi_5 + -(-i)\xi_6 + i\xi_7 + i\xi_8 - 2\frac{1}{2}(-\sqrt{3} + i)\xi_9 \right. \\ &\quad \left. + 2\frac{1}{2}(-i - \sqrt{3})\xi_{10} \right), \\ i \operatorname{Sh}_{F^2/F}(\tilde{\theta}_{11}) &= \frac{\sqrt{3}}{6} (-\sqrt{3}(-i)\xi_5 + \sqrt{3}(-i)\xi_6 + \sqrt{3}i\xi_7 - \sqrt{3}i\xi_8). \end{aligned}$$

We set  $u_1 = u_2 = u_{10} = 1$  and  $u_{11} = i$ . If we compare the coefficients in the preceding computations with the coefficients of the Fourier matrix of Geck-Malle recalled in §1, we obtain up to a sign a row of the Fourier matrix. Therefore the conjecture holds.  $\square$

**Remark 5.2.** We now discuss the roots  $u_i$  for  $i \in \{1, 2, 10, 11, 12[1], 12[-1]\}$  appearing in the proof of Theorem 5.1.

For  $g \in \mathbf{G}^F$ , Lang's theorem says that there is  $x \in \mathbf{G}$  such that  $g = x^{-1}F(x)$ . Therefore we have  $xgx^{-1} \in \mathbf{G}^F$ . Moreover, if  $x' \in \mathbf{G}$  is such that  $x'^{-1}F(x') = g$ , then  $x'$  lies in the coset  $\mathbf{G}^F.x$ , hence the  $\mathbf{G}^F$ -class of  $xgx^{-1}$  does not depend on the choice of  $x$ . For a class function  $f$  on  $\mathbf{G}^F$ , we can then define the Asai-twisting operator  $t^*$  by

$$t^*(f)(g) = f(xgx^{-1}) \quad \text{for } g \in \mathbf{G}^F, \text{ and } x \in \mathbf{G} \text{ such that } x^{-1}F(x) = g.$$

For a pair  $(g, \psi)$  with  $g \in \mathbf{G}^F$  and  $\psi$  an  $F$ -stable irreducible character of the component group  $A(g) = \mathbf{C}_{\mathbf{G}}(g)/\mathbf{C}_{\mathbf{G}}(g)^\circ$ , we can associate a class function  $\Psi_{(g, \psi)}$  which depends on the choice of an extension of  $\psi$  to  $A(g) \rtimes \langle F \rangle$ . Therefore  $\Psi_{(g, \psi)}$  is an eigenvector of  $t^*$ , and the corresponding eigenvalue  $\lambda_{(g, \psi)}$  is equal to  $\psi(\bar{g})/\psi(1)$ , where  $\bar{g}$  denotes the image of  $g$  in  $A(g)$ .

Let  $(c_i, i \in I)$  be a row of a Fourier matrix. Then the construction of the Fourier matrices given by Geck-Malle show that there is a pair  $(g, \psi)$  as above, such that

$$\Psi_{(g, \psi)} = \sum_{i \in I} c_i \xi_i.$$

To  $\chi$  an  $F$ -stable unipotent character of  $\mathbf{G}^{F^2}$ , we associate the pair  $(g, \psi)$  attached as above to the row of the Fourier matrix corresponding to  $u_\chi \operatorname{Sh}_{F^2/F}(\tilde{\chi})$ .

We have

$\chi$	$\lambda_{(g,\psi)}$
$\theta_1$	1
$\theta_2$	1
$\theta_{10}$	1
$\theta_{11}$	-1
$\theta_{12[1]}$	$\frac{1}{2}(-1 - \sqrt{3}i)$
$\theta_{12[-1]}$	$\frac{1}{2}(-1 + \sqrt{3}i)$

We observe that the element  $u_\chi$  chosen in the proof of Theorem 5.1 is a root of the polynomial

$$X^2 - \lambda_{(g,\psi)}.$$

Moreover, we remark that we can choose for  $u_\chi$  an arbitrary root of this polynomial, because a row of a Fourier matrix is defined up to a sign.

Finally, we notice that in the situation of type  $B_2$  with  $F$  the Frobenius map defining the Suzuki groups, this observation also holds [1, 4.2].

**Remark 5.3.** Let  $\mathbf{G}$  be a simple algebraic group of type  $F_4$  over  $\overline{\mathbb{F}}_2$ , and let  $F$  be the Frobenius map on  $\mathbf{G}$  that defines the Ree group  ${}^2F_4(2^{2n+1})$ . In this situation, the character table of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  is actually unknown. Suppose that

- The conjecture of Digne-Michel holds.
- We know how to associate to every unipotent character of  ${}^2F_4(q)$  its root of unity as above.
- The observation of Remark 5.2 holds.

Then, using [7, 5.4(c)] and [10], we can describe the values of the unipotent characters of  $\mathbf{G}^{F^2} \rtimes \langle F \rangle$  on the coset  $\mathbf{G}^{F^2}.F$ .

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